PH-107 (2021): Tutorial Sheet 1

* marked problems will be solved in the Wednesday tutorial class

Photoelectric Effect:

1. *In a photoelectric effect experiment, excited hydrogen atoms are used as light source. The light emitted from this source is directed to a metal of work function Φ. In this experiment, the following data on stopping potentials (V_s), for various Balmer lines of hydrogen, is obtained.

$$n = 4 \to n = 2, \text{ transition line : } V_S = 0.43 \text{ V}$$
$$n = 5 \to n = 2, \text{ transition line : } V_S = 0.75 \text{ V}$$
$$n = 6 \to n = 2, \text{ transition line : } V_S = 0.94 \text{ V}$$

   a) What is the work function Φ of the metal in eV?
   b) What is the stopping potential (in Volts) for Balmer line of the shortest wavelength?
   c) What will be the photocurrent corresponding to Paschen series (ending in $n = 3$) transitions?

2. In an experiment on photoelectric effect of a metal, the stopping potentials were found to be 4.62 V and 0.18 V for $\lambda_1 = 1850$ Å and $\lambda_2 = 5460$ Å, respectively. Find the value of Planck's constant, the threshold frequency and the work function of the metal.

3. A monochromatic light of intensity 1.0 μW/cm^2 falls on a metal surface of area 1 cm^2 and work function 4.5 eV. Assume that only 3% of the incident light is absorbed by the metal (rest is reflected back) and that the photoemission efficiency is 100 % (i.e. each absorbed photon produces one photo-electron). The measured saturation current is 2.4 nA.

   (a) Calculate the number of photons per second falling on the metal surface.
   (b) What is the energy of the incident photon in eV ?
   (c) What is the stopping potential ?

4. In a photoelectric experiment, a photocathode is illuminated separately by two light sources of same intensity but different wavelengths, 480 nm and 613 nm. The resulting photocurrent is measured as a function of the potential difference (V) between the cathode and the anode. Observed photocurrent for three values of V is given below

V	current (nA)	
	480 nm	613 nm
−0.1	76.3097	64.7039
−0.2	67.6194	44.4078
−0.3	58.9291	24.1118

   (a) Using this data, obtain the work function of the photocathode and the cut off wavelength.
   (b) What is the maximum kinetic energy of the electron for $\lambda = 480$ nm? What should be the wavelength of light to emit electrons half this kinetic energy?

(c) When the photocathode material is changed, it is found that the cut off frequency is 1.2 times the cut off frequency of the old material. What is the work function of the new material?

5. Light of wavelength 2000 Å falls on a metal surface. If the work function of the metal is 4.2 eV, find the kinetic energy of the fastest and the slowest emitted photoelectrons. Also find the stopping potential and cutoff wavelength for the metal.

Black Body Radiation:

1. * According to Planck, the spectral energy density $u(\lambda)$ of a blackbody maintained at temperature T is given by

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}$$

where λ denotes the wavelength of radiation emitted by the blackbody.

(a) Find an expression for $\lambda_{\max}$ at which $u(\lambda, T)$ attains its maximum value (at a fixed temperature T). $\lambda_{\max}$ should be in terms of T and fundamental constants h, c and k_B.

(b) Expressing $\lambda_{\max}$ as $\frac{\alpha}{T}$, obtain an expression for $u_{\max}(T)$ in terms of α, T and the fundamental constants.

2. The earth rotates in a circular orbit about the sun. The radius of the orbit is 140×10^6 km. The radius of the earth is 6000 km and the radius of the sun is 700,000 km. The surface temperature of the sun is 6000 K. Assuming that the sun and the earth are perfect black bodies, calculate the equilibrium temperature of the earth.

3. (a) Given Planck's formula for the energy density, obtain an expression for the Rayleigh Jeans formula for $U(\nu, T)$.

   (b) For a black body at temperature T, $U(\nu, T)$ was measured at $\nu = \nu_0$. This value is found to be one tenth of the value estimated using Rayleigh Jeans formula. Obtain an implicit equation in terms of $h\nu/k_B T$

   (c) Solve the above equation to obtain the value of $h\nu/k_B T$, up to the first decimal place.

4. Using appropriate approximations, derive Weins' displacement law from Planck's formula for energy density of black body radiation.

Compton Scattering:

1. A photon of energy $h\nu$ is scattered through 90° by an electron initially at rest. The scattered photon has a wavelength twice that of the incident photon. Find the frequency of the incident photon and the recoil angle of the electron.

2. Find the energy of the incident x-ray if the maximum kinetic energy of the Compton electron is $m_0 c^2/2.5$.

3. Show that a free electron cannot absorb a photon so that a photoelectron requires bound electron. However, the electron can be free in Compton Effect. Why?

4. Two Compton scattering experiments were performed using x-rays (incident energies E_1 and $E_2 = E_1/2$). In the first experiment, the increase in wavelength of the scattered x-ray, when measured at an angle $\theta = 45°$, is 7×10^{-14} m. In the second experiment, the wavelength of the scattered x-ray, when measured at an angle $\theta = 60°$, is 9.9×10^{-12} m.

   (a) Calculate the Compton wavelength and the mass (m) of the scatterer.

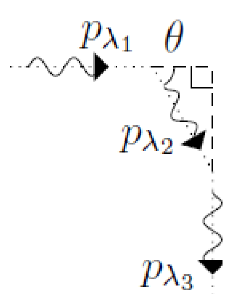
   (b) Find the wavelengths of the incident x-rays in the two experiments.

5. Find the smallest energy that a photon can have and still transfer 50% of its energy to an electron initially at rest.

6. *γ-rays are scattered from electrons initially at rest. Assume the it is back-scattered and its energy is much larger than the electron's rest-mass energy, $E \gg m_e c^2$.
   (a) Calculate the wavelength shift
   (b) Show that the energy of the scattered beam is half the rest mass energy of the electron, regardless of the energy of the incident beam
   (c) Calculate the electron's recoil kinetic energy if the energy of the incident radiation is 150MeV

7. In Compton Scattering, show that the maximum energy of the scattered photon will be $2m_0c^2$, irrespective of the energy of the incident photon. Find the value of θ_0, the angle at which the maximum energy occurs.

8. * In a Compton scattering experiment (see figure), X-rays scattered off a free electron initially at rest at an angle $\theta(> \pi/4)$), gets re-scattered by another free electron, also initially at rest.



   (a) If $\lambda_3 - \lambda_1 = 1.538 \times 10^{-12}$ m, find the value of θ.

   (b) If $\lambda_2 = 68 \times 10^{-12}$ m , find the angle at which the first electron recoils due to the collision.

PH-107 (2021): Tutorial Sheet 2

* marked problems will be solved in the Wednesday tutorial class

de Broglie Wavelength:

1. Calculate the wavelength of the matter waves associated with the following:

   (a) A 2000 kg car moving with a speed of 100 km/h.

   (b) A 0.28 kg cricket ball moving with a speed of 40 m/s.

   (c) An electron moving with a speed of 10^7 m/s.

   Compare in each case the result with the respective dimension of the object. In which case will it be possible to observe the wave nature.

2. Show that the Bohr's angular momentum quantization leads to the formation of standing waves by the electrons along the orbital circumference in hydrogen atom.

3. Calculate the de-Broglie wavelength (in nm) for a photon, an electron and a neutron each with an energy of 5 keV (for electron and neutron, the energy refers to non-relativistic kinetic energy). Take $m_e = 500$ keV/c^2 and $m_n = 1000$ MeV/c^2.

4. *Thermal kinetic energy of a hydrogen atom is $\sim k_B T$ and the radius is $\sim r_1$ (= 0.53 Å, radius of the $n = 1$ Bohr orbit). Find the temperature at which its de Broglie wavelength has a value of $2r_1$. Take the mass of the hydrogen atom to be that of a proton.

Interference, Diffraction, YDSE, Davison-Germer experiment :

1. * Buckminsterfullerene are soccer-like balls (called buckyballs) made up of 60 carbon atoms (C_{60}). A double slit experiment is performed using these buckyballs travelling at a velocity of 100 m/sec (slit width = 150 nm and the separation between the slits and the screen, $D = 1.25$ m from the slits).

   (a) Find the de Broglie wavelength of the buckyball.

   (b) Find the distance between the maxima of the resultant interference pattern. Treat the buck balls as point like objects.

   (c) The size of the buckyballs is ∼10 Å. How does the size of the ball compare with the distance between the neighboring maxima of the interference patterns? Is the size of C_{60} likely to affect the visibility of the interference fringes? Find the initial velocity of C_{60} for which the interference fringes start to become difficult to detect?

2. Consider two plane waves, one with a wave vector, $\vec{k}_1 = (2\pi/\lambda)(\vec{x} + \vec{y} + \vec{z})$, and the other with $\vec{k}_2 = (2\pi/\lambda)\vec{z}$. For $\lambda = 500$ nm, (a) find the resultant wave due to the interference of these two waves, (b) calculate the intensity and (c) analyze the interference pattern in the xy-plane, i.e. the condition for maxima and minima.

3. In a Young's double slit experiment, the slit separation is 0.8 mm and the observing plane is 1.6 m away from the two slits.

   (a) Plot the intensity pattern at the observing plane.

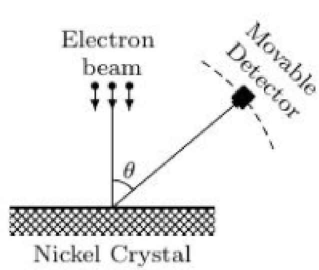
   (b) If the distance between the two consecutive maxima is 5 mm, find the wavelength of the light.

   (c) When one of the slits is covered by a transparent thin film, the central maximum is seen to shift by 2.2 fringes. If the refractive index of the film is 1.4, find the thickness of the film.

   (d) Now the two slits are illuminated by a light containing two wavelengths, 450 nm and 600 nm. What is the least order at which a maximum of one wavelength will fall exactly on a minimum of the other?

4. *In a double-slit experiment with a source of monoenergetic electrons, detectors are placed along a vertical screen parallel to the y-axis to monitor the diffraction pattern of the electrons emitted from the two slits. When only one slit is open, the amplitude of the electrons detected on the screen is $\psi_1(y,t) = A_1 e^{-i(ky-\omega t)}/\sqrt{1+y^2}$, and when only the other is open the amplitude is $\psi_2(y,t) = A_2 e^{-i(ky+\pi y-\omega t)}/\sqrt{1+y^2}$, where A_1 and A_2 are normalization constants. Calculate the intensity detected on the screen when
   (a) both slits are open and a light source is used to determine which of the slits the electron went through and
   (b) both slits are open and no light source is used.
   (c) Plot the intensity registered on the screen as a function of y for cases (a) and (b).

5. *In a Davisson-Germer experiment, electrons having energy of 54eV were bombarded normally over copper crystal. The diffracted beam was recorded using a detector and when the intensity of the diffracted electrons was plotted against the angle with the normal of the surface and the 1st maxima was observed at an angle of $\theta = 35°$.



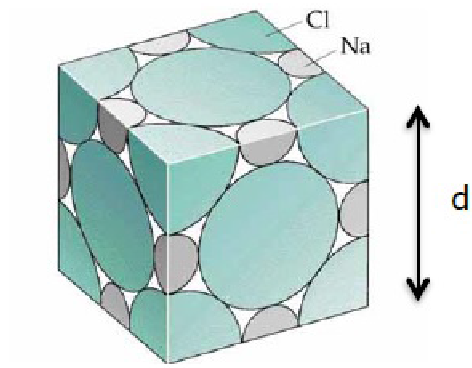
(a) Calculate the spacing between the atoms on the copper surface.

(b) What other angles are possible for a maxima?

(c) If the energy of incident electrons were increased by 3 times. Find the location of first maxima.

(d) How many more intensity peaks (maxima's) will be observed as the angle is further increased?

6. Sodium Chloride (NaCl) crystal is made up of cubes of edge length d, as shown in the figure. Each cube contains a full Na ion at its body center, which is not shown in the figure. In a Davisson-Germer experiment, performed using electrons of kinetic energy 40 eV, the NaCl crystal gives a first order ($n = 1$) diffraction peak at 20.11°.



(a) Compute d

(b) Compute the number of NaCl molecules in the given cube.

(c) Given the density and the molecular weight of NaCl to be 2.17 g/cm^3 and 58.44 g/mol, respectively, compute Avogadro's number.

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Tutorial Sheet 3

* marked problems will be solved in the Wednesday tutorial class

Wave packets: Group and Phase Velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

where x, y are in meters and t is in sec.

   (a) Find the resultant wave and its phase and group velocities

   (b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ? [Ans.: $v_p = 50$ m/s, $v_g = 50$ m/s, $\Delta x = 5\pi$ m, $\Delta x \Delta k = 2\pi$]

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
(a) very large wavelengths and
(b) very small wavelengths.

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

   (a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ε and permeability μ.

   (b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

6. The dispersion relation for a lattice wave propagating in a 1-D chain of atoms of mass m bound together by a force constant β is given by $\omega = \omega_0 \sin\left(\frac{ka}{2}\right)$, where $\omega_0 = \sqrt{4\beta/m}$ and a is the distance between the atoms.

   (a) Show that the medium is non- dispersive in the long wavelength limits.

   (b) Find the group and phase velocities at $k = \pi/a$. [Ans.: 0, $\omega_o a/\pi$]

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x (1 - \cos k_x a_x) + 2\beta_y (1 - \cos k_y a_y) = 0$$

where $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis

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Tutorial Sheet 4

Only "*" to be solved in the tutorials

Fourier Transform

1. *If $\phi(k) = A(a - |k|), |k| \leq a$, and 0 elsewhere. Where a is a positive parameter and A is a normalization factor to be found.
   (a) Find the Fourier transform for $\phi(k)$
   (b) Calculate the uncertainties Δx and Δp and check whether they satisfy the uncertainty principle.

2. A wave packet is of the form $f(x) = \cos^2\left(\frac{x}{2}\right)(for - \pi \leq x \leq \pi)$ and $f(x) = 0$ elsewhere
   (a) Plot $f(x)$ versus x.
   (b) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$?
   (c) At what value of $k, |g(k)|$ attains its maximum value?
   (d) Calculate the value(s) of k where the function $g(k)$ has its first zero.
   (e) Considering the first zero(s) of both the functions $f(x)$ and $g(k)$ to define their spreads (i.e. Δx and Δk), calculate the uncertainty product $\Delta x.\Delta k$.

3. *A wave function $\psi(x)$ is defined such that $\psi(x) = \sqrt{2/L}\sin(\pi x/L)$ for $0 \leq x \leq L$ and $\psi(x) = 0$ otherwise.
   (a) Writing $\psi(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$, find $a(k)$.
   (b) What is the amplitude of the plane wave of wavelength L constituting $\psi(x)$?

4. A wave packet is of the form $f(x) = \exp(-\alpha|x|) \cdot \exp(ik_0 x)$ (for $-\infty \leq x \leq \infty$) where α, k_0 are positive constants.
   (a) Plot $|f(x)|$ versus x.
   (b) At what values of x does $|f(x)|$ attain half of its maximum value? Consider the full width at half maxima (FWHM) as a measure of the spread (uncertainty) in x, find Δx
   (c) Calculate the Fourier transform of $f(x)$, i.e. $g(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}dx$
   (d) Plot $g(k)$ versus k.
   (e) Find the values of k at which $g(k)$ attains half its maximum value? Using the same concept of FWHM as in part (b), calculate Δk ? Hence calculate the product $\Delta x.\Delta k$
   $\left[\text{ Given : } \int_0^{\infty} e^{-(\alpha - ik)x}dx = \frac{1}{\alpha - ik}\right]$

Heisenberg Uncertainty Principle

1. Estimate the uncertainty in the position of (a) a neutron moving at 5×10^6 m s^{-1} and (b) a 50 kg person moving at 2 m s^{-1}.

2. A lead nucleus has a radius 7×10^{-15} m. Consider a proton bound within nucleus. Using the uncertainty relation $\Delta \mathrm{p}.\Delta \mathrm{r} \geq \hbar/2$, estimate the root mean square speed of the proton, assuming it to be non-relativistic. (You can assume that the average value of p^2 is square of the uncertainty in momentum.)

3. For a non-relativistic electron, using the uncertainty relation $\Delta x \Delta p_x = \hbar/2$
   (a) Derive the expression for the minimum kinetic energy of the electron localized in a region of size ' a '.
   (b) If the uncertainty in the location of a particle is equal to its de Broglie wavelength, show that the uncertainty in the measurement of its velocity is same as the particle velocity.
   (c) Using the expression in (b), calculate the uncertainty in the velocity of an electron having energy 0.2keV
   (d) An electron of energy 0.2keV is passed through a circular hole of radius 10^{-6} m. What is the uncertainty introduced in the angle of emergence in radians? (Given $\tan\theta \cong \theta$)

4. A particle of mass m moves in a one-dimensional potential $V(x) = \alpha|x|$ where $\alpha > 0$. Using Heisenberg's uncertainty relation, the minimum total energy of the particle is found to have the form $E_{\text{min}} = AB^{1/3}$. Find A and B.

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Tutorial Sheet 5

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Operators and Wave function

1. Which of the operators A_i defined in the following are linear operators? Which of these are hermitian? All the functions $\psi(x)$ are well behaved functions vanishing at $\pm\infty$.
   (a) $\hat{A}_1\psi(x) = \psi(x)^2$
   (b) $\hat{A}_2\psi(x) = \frac{\partial\psi(x)}{\partial x}$
   (c) $\hat{A}_3\psi(x) = \int_a^x \psi(x')\, dx'$
   (d) $\hat{A}_4\psi(x) = 1/\psi(x)$
   (e) $\hat{A}_5\psi(x) = -\psi(x+a)$
   (f) $\hat{A}_6\psi(x) = \sin(\psi(x))$
   (g) $\hat{A}_7\psi(x) = \frac{\partial^2\psi(x)}{\partial x^2}$

2. (a) If $\hat{A}$ and $\hat{B}$ are Hermitian and $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$, prove that $\hat{C}$ is Hermitian
   (b) An operator is said to be anti-Hermitian if $\hat{O}^\dagger = -\hat{O}$. Prove that $[\hat{A},\hat{B}]$ is anti-Hermitian.

3. * Prove that if $\hat{K}$ is a Hermitian operator, $\exp\left(i\hat{K}\right)$ is an unitary operator, and if $\hat{U}$ is an Unitary operator, then there is an operator K such that $\hat{U} = \exp\left(i\,\hat{K}\right)$, and this $\hat{K}$ is Hermitian.

4. If $\hat{A}$ and $\hat{B}$ are operators, prove
   (a) that $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$
   (b) that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$
   (c) that $\hat{A}+\hat{A}^\dagger, i\left(\hat{A}-\hat{A}^\dagger\right)$, and that $\hat{A}\hat{A}^\dagger$ are Hermitian operators.

5. An operator is given by
$$\hat{G} = i\hbar\frac{\partial}{\partial x} + Bx$$
   where B is a constant. Find the eigen function $\phi(x)$. If this eigen function is subjected to a boundary condition $\phi(a) = \phi(-a)$ find out the eigen values.

6. $\Psi_1(x)$ and $\Psi_2(x)$ are the normalized eigen functions of an operator $\hat{P}$, with eigen values P_1 and P_2 respectively. If the wave function of a particle is $0.25\Psi_1(x) + 0.75\Psi_2(x)$ at $t = 0$, find the probability of observing P_1.

7. * Consider a large number (N) of identical experimental set-ups. In each of these, a single particle is described by a wave function $\Phi(x) = A\exp(-x^2/b^2)$ at $t = 0$, where A is the normalization constant and b is another constant with the dimension of length. If a measurement of the position of the particle is carried out at time $t = 0$ in all these set-ups, it is found that in 100 of these, the particle is found within an infinitesimal interval of $x = 2b$ to $2b + dx$. Find out, in how many of the measurements, the particle would have been found in the infinitesimal interval of $x = b$ to $b + dx$.

8. * An observable A is represented by the operator $\hat{A}$. Two of its normalized eigen functions are given as $\Phi_1(x)$ and $\Phi_2(x)$, corresponding to distinct eigenvalues a_1 and a_2, respectively. Another observable B is represented by an operator $\hat{B}$. Two normalized eigen functions of this operator are given as $u_1(x)$ and $u_2(x)$ with distinct eigenvalues b_1 and b_2, respectively. The eigen functions $\Phi_1(x)$ and $\Phi_2(x)$ are related to $u_1(x)$ and $u_2(x)$ as, $\Phi_1 = D(3u_1 + 4u_2)$; $\Phi_2 = F(4u_1 - Pu_2)$ At time t = 0, a particle is in a state given by $\frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$.

(a) Find the values of D, F and P.
(b) If a measurement of A is carried out at t = 0, what are the possible results and what are their probabilities ?
(c) Assume that the measurement of A mentioned above yielded a value a1. If a measurement of B is carried out immediately after this, what would be the possible outcomes and what would be their probabilities ?
(d) If instead of following the above path, a measurement of B was carried out initially at t = 0, what would be the possible outcomes and what would be their probabilities ?
(e) Assume that after performing the measurements described in (c), the outcome was b_2. What would be the possible outcomes, if A were measured immediately after this and what would be the probabilities ?

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Tutorial Sheet 6

* marked problems will be solved in the Wednesday tutorial class

Free particle

1. *Show that

$$\psi(x) = A\,\sin(kx) + B\,\cos(kx)$$

and

$$\psi(x) = Ce^{ikx} + De^{-ikx}$$

are equivalent solutions of TISE of a free particle. A, B, C and D can be complex numbers.

2. Show that

$$\Psi(x,t) = A\sin(kx-\omega t) + B\cos(kx-\omega t)$$

does not obey the time-dependant Schroedinger's equation for a free particle.

3. The wave function for a particle is given by,

$$\phi(x) = Ae^{ikx} + Be^{-ikx}$$

where A and B are real constants. Show that $\phi(x)^*\phi(x)$ is always a positive quantity.

4. * A free proton has a wave function given by

$$\Psi(x,t) = Ae^{i(5.02*10^{11}x - 8:00*10^{15}t)}$$

The coefficient of x is inverse meters, and the coeffcient of t is inverse seconds. Find its momentum and energy.

5. A particle moving in one dimension is in a stationary state whose wave function,

$$\Psi(x) = \begin{cases} 0, & \text{x < -a} \\ A\left(1+\cos\frac{\pi x}{a}\right), & -a \le x \le a \\ 0, & \text{x > a} \end{cases}$$

where A and a are real constants.

(a) Is this a physically acceptable wave function? Explain.

(b) Find the magnitude of A so that $\psi(x)$ is normalized.

(c) Evaluate Δx and Δp. Verify that $\Delta x \Delta p \geq \hbar/2$.

(d) Find the classically allowed region.

6. * Consider the 1-dimensional wave function of a particle of mass m, given by

$$\psi(x) = A \left(\frac{x}{x_0} \right)^n e^{-\frac{x}{x_0}}$$

where, A, n and x_0 are real constants.

(a)Find the potential $V(x)$ for which $\psi(x)$ is a stationary state (It is known that $V(x) \to 0$ as $x \to \infty$).

(b)What is the energy of the particle in the state $\psi(x)$?

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Tutorial Sheet 7

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Particle in a Box:

1. * For a particle in a 1-D box of side L, show that the probability of finding the particle between $x = a$ and $x = a + b$ approaches the classical value b/L, if the energy of the particle is very high.

2. Consider a particle confined to a 1-D box. Find the probability that the particle in its ground state will be in the central one-third region of the box.

3. Consider a particle of mass m moving freely between $x = 0$ and $x = a$ inside an infinite square well potential.
(a) Calculate the expectation values $\langle \hat{X} \rangle_n$, $\langle \hat{P} \rangle_n$, $\left\langle \hat{X}^2 \right\rangle_n$, and $\left\langle \hat{P}^2 \right\rangle_n$, and compare them with their classical counterparts.
(b) Calculate the uncertainties product $\Delta x_n \Delta p_n$.
(c) Use the result of (b) to estimate the zero-point energy.

4. Consider a one dimensional infinite square well potential of length L. A particle is in $n = 3$ state of this potential well. Find the probability that this particle will be observed between $x = 0$ and $x = L/6$. Can you guess the answer without solving the integral? Explain how.

5. * Consider a one-dimensional particle which is confined within the region $0 \leq x \leq a$ and whose wave function is $\Psi(x,t) = \sin(\pi x/a)\exp(-i\omega t)$.
(a) Find the potential $V(x)$.
(b) Calculate the probability of finding the particle in the interval $a/4 \leq x \leq 3a/4$.

6. An electron is moving freely inside a one-dimensional infinite potential box with walls at $x = 0$ and $x = a$. If the electron is initially in the ground state ($n = 1$) of the box and if we suddenly quadruple the size of the box (i.e., the right-hand side wall is moved instantaneously from $x = a$ to $x = 4a$), calculate the probability of finding the electron in:
(a) the ground state of the new box and
(b) the first excited state of the new box.

7. Solve the time independent Schrodinger equation for a particle in a 1-D box, taking the origin at the centre of the box and the ends at $\pm L/2$, where L is the length of the box.

8. * Consider a particle of mass m in an infinite potential well extending from $x = 0$ to $x = L$. Wave function of the particle is given by

$$\psi(x) = A\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

where A is the normalization constant

(a) Calculate A

(b) Calculate the expectation values of x and x^2 and hence the uncertainty Δx.

(c) Calculate the expectation values of p and p^2 and hence the uncertaintyΔp.

(d) What is the probability of finding the particle in the first excited state, if an energy measurement is made?

(given, $\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = 0$, $\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = 0$, for all n)

9. Suppose we have 10,000 rigid boxes of same length L from $x = 0$ to $x = L$. Each box contains one particle of mass m. All these particles are in the ground state.

(a) If a measurement of position of the particle is made in all the boxes at the same time, in how many of them, the particle is expected to be found between $x = 0$ and $L/4$?

(b) In a particular box, the particle was found to be between $x = 0$ and $L/4$. Another measurement of the position of the particle is carried out in this box immediately after the first measurement. What is the probability that the particle is again found between $x = 0$ and $L/4$?

10. * An electron is bound in an infinite potential well extending from $x = 0$ to $x = L$. At time $t = 0$, its normalized wave function is given by

$$\psi(x,0) = \frac{2}{\sqrt{L}}\sin\left(\frac{3\pi x}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)$$

(a) Calculate $\psi(x,t)$ at a later time t.

(b) Calculate the probability of finding the electron between $x = L/4$ and $x = L/2$ at time t.

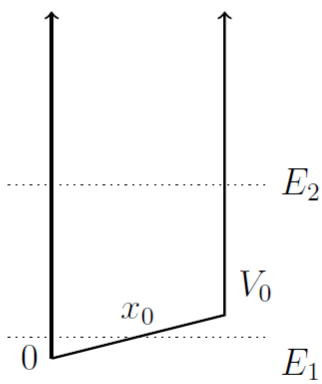
11. A speck of dust ($m = 1$ μg) is trapped to roll inside a tube of length $L = 1.0$ μm. The tube is capped at both ends and the motion of the speck is considered to be along the length of the tube.

(a) Modeling this as a 1-D infinite square well, determine the value of the quantum number n if the speck has an energy of 1 μJ.

(b) What is the probability of finding this speck within 0.1 μm of the center of the tube ($0.45 < x < 0.55$).

(c) How much energy is needed to excite this speck to an energy level next to 1 μJ? Compare this excitation energy with the thermal energy at room temperature ($T = 300$ K).

12. Consider a particle bound inside an infinite well whose floor is sloping (variation is small) as shown in the figure. Without solving the Schrodinger equation (provide proper justification for your answers),



(a) sketch a plausible wave function when the energy is E_1, assuming that it has no nodes.

(b) Sketch the wave function with 5 nodes when the energy is E_2.

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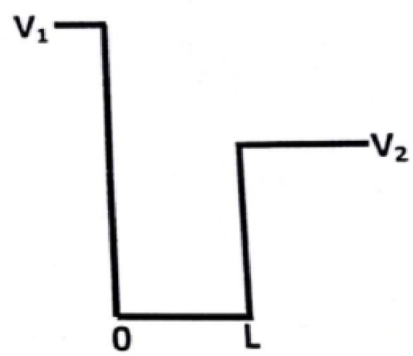
* marked problems will be solved in the Wednesday tutorial class

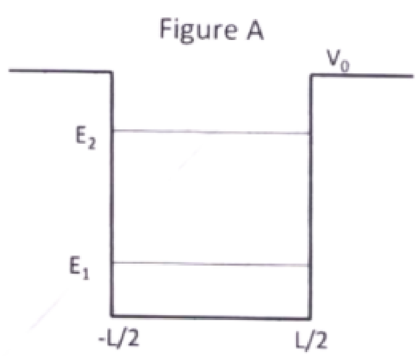
Particle in a Finite Box:

1. * Consider the asymmetric finite potential well of width L, with a barrier V_1 on one side and a barrier V_2 on the other side. Obtain the energy quantization condition for the bound states in such a well. From this condition derive the energy quantization conditions for a semi-infinite potential well (when $V_1 \to \infty$ and V_2 is finite).



2. Consider a particle of mass 'm' trapped in a finite square box of length 'a' with barrier height equals to V_0. Find the number of bound states and the corresponding energies for the finite square well potential if $\sqrt{ma^2V_0/\left(2\hbar^2\right)} = 1$).

3. An electron is trapped in a 1-dimensional symmetric potential well of height V_0 and width L (Figure A). The energy of electron is E, its wave number inside the well is k, and the magnitude of the wave number outside the well is α.



(a) Derive the energy quantization conditions, in terms of k, α and L, for the symmetric and anti-symmetric bound states.

(b) When the width of the well is L = 0.2 nm, it is found that the ground state energy $E_1 = 4.45$eV, and the first excited state energy $E_2 = 15.88$eV. Calculate V_0.

(c) Calculate the penetration depth for the ground state.

(d) If the width of the potential well is doubled to 2 L keeping V_0 the same, estimate the change in the ground state energy.

(e) Consider the potential which is generated from Figure A by setting $V = \infty$ at L = 0. What is the energy of the ground state in this case?

(f) How many bound states are possible in this case?

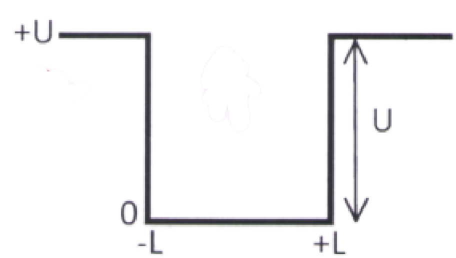
4. *Consider a particle of mass m in a potential given by

$$\begin{aligned} V(x) &= 0 \text{ for } |x| < L/2, \\ &= V_0 \text{ for } L/2 < |x| < L \\ &= \infty \text{ for } |x| \geq L \end{aligned}$$

(a) Sketch the potential and the qualitative nature of the ground-state wave-function (without solving the Schrodinger equation). Mention the functional form of the wave function in each region.

(b) An electron is trapped in a symmetric finite potential of depth $V_0 = 1000$eV and width $L = 1$. What is approximate energy of the ground state?

5. A particle with energy E is bound in a finite square well potential with height U and width $2L$ (as shown in the figure below)

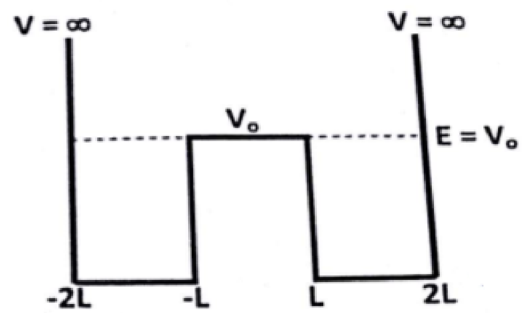


(a) Consider the case $E < U$, obtain the energy quantization condition for the symmetric wave functions in terms of K and α, where $K = \sqrt{2mE/\hbar^2}$ and $\alpha = \sqrt{2m(U-E)/\hbar^2}$

(b) Apply this result to an electron trapped at a defect site in a crystal. Modeling the defect as a finite square well potential with height 5eV and width 200pm, calculate the ground state energy ?

(c) Calculate the total number of bound states with symmetric wavefunction?

6. *A particle of mass m is bound in a double well potential shown in the figure. Itsenergy eigenstate $\psi(x)$ has energy eigenvalue $E = V$ o (where V_o is the energy of the plateau in the middle of the potential well). It is known that $\psi(x) = C$ (C is a constant) in the plateau region.



(a) Obtain $\psi(x)$ for the regions -2 L $<$ x $<$ $-$L and $L < x < 2L$ and the relation between the wavenumber ' k ' and L.

(b) Determine ' C ' in terms of L.

(c) Assume that the bound particle is an electron and $L = 1A$. Calculate the 2 lowest values of V_o (in eV) for which such a solution exists.

(d) For the smallest allowed k, calculate the expectation values for x, x^2, p and p^2 and show that Heisenberg's Uncertainty Relation is obeyed.

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* marked problems will be solved in the tutorial class (D3-D4: Wednesday, D1-D2: Saturday)

Scattering problems:

1. * A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy E ($< V_0$) approaches the barrier from left.

   (a) Find the value of $x = x_0$ ($x_0 > 0$), for which the probability density is $1/e$ times the probability density at $x = 0$.

   (b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.

2. Consider a potential

$$\begin{aligned} V(x) &= 0 \text{ for } x < 0, \\ &= -V_0 \text{ for } x > 0 \end{aligned}$$

   Consider a beam of non-relativistic particles of energy $E > 0$ coming from $x \to -\infty$ and being incident on the potential. Calculate the reflection and transmission coefficients.

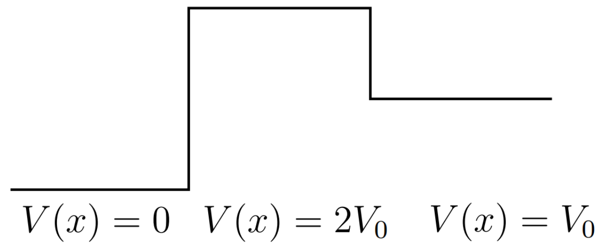
3. A potential barrier is defined by $V = 0$ eV for $x < 0$ and $V = 7$ eV for $x > 0$. A beam of electrons with energy 3 eV collides with this barrier from left. Find the value of x for which the probability of detecting the electron will be half the probability of detecting it at $x = 0$.

4. * A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.

   (a) Obtain an expression for the transmission coefficient.

   (b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.

5. A beam of particles of energy $E < V_0$ is incident on a barrier (see figure below) of height $V = 2V_0$. It is claimed that the solution is $\psi_I = A\exp(-k_1 x)$ for region I ($0 < x < L$) and $\psi_{II} = B\exp(-k_2 x)$ for region II ($x > L$), where $k_1 = \sqrt{\frac{2m(2V_0-E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$. Is this claim correct? Justify your answer.

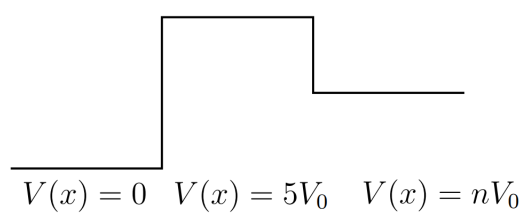


6. * A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0$, $V = 5V_0$ for $x \leq d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \pi h/\sqrt{8mV_0}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.

   (a) Find n. Are there more than one possible values for n?

   (b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

   (c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used



7. A scanning tunneling microscope (STM) can be approximated as an electron tunneling into a step potential [$V(x) = 0$ for $x \leq 0$, $V(x) = V_0$ for $x > 0$]. The tunneling current (or probability) in an STM reduces exponentially as a function of the distance from the sample. Considering only a single electron-electron interaction, an applied voltage of 5 V and the sample work function of 7 eV, calculate the amplification in the tunneling current if the separation is reduced from 2 atoms to 1 atom thickness (take approximate size of an atom to be 3 Å).

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Tutorial Sheet 10

* marked problems will be solved in the tutorial class (D3-D4: Wednesday, D1-D2: Saturday)

Simple Harmonic Oscillator and 2D/3D Systems:

1. Using the uncertainty principle, show that the lowest energy of an oscillator is $\hbar\omega/2$.

2. Determine the expectation value of the potential energy for a quantum harmonic oscillator (with mass m and frequency ω) in the ground state. Use this to calculate the expectation value of the kinetic energy. The ground state wavefunction of quantum harmonic oscillator is:
$$\psi_0(x) = C_0 \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad C_0 \text{ is constant;} \tag{1}$$

3. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant $k = 12Nm^{-1}$ and mass $m = 5.6 * 10^{-26}kg$
   (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state ?
   (b) Find the ground state energy of vibrations for this diatomic molecule.

4. Vibrations of the hydrogen molecule can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 * 10^3 Nm^{-2}$ and mass $m = 1.67 * 10^{27}$ kg.
   (a) What is the vibrational frequency of this molecule ?
   (b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states ?

5. * A two-dimensional isotropic harmonic oscillator has the Hamiltonian
$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}k(x^2 + y^2)$$
   (a) Show that the energy levels are given by
$$E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1) \quad \text{where} \quad n_x, n_y \in (0, 1, 2...) \quad \omega = \sqrt{\frac{k}{m}}$$
   (b) What is the degeneracy of each level?

6. * Consider the Hamiltonian of a two-dimensional anisotropic harmonic oscillator ($\omega_1 \neq \omega_2$)
$$H = \frac{{p_1}^2}{2m} + \frac{{p_2}^2}{2m} + \frac{1}{2}m{\omega_1}^2{q_1}^2 + \frac{1}{2}m{\omega_2}^2{q_2}^2$$

(a) Exploit the fact that the Schrödinger eigenvalue equation can be solved by separating the variables and find a complete set of eigenfunctions of H and the corresponding eigenvalues.

(b) Assume that $\frac{\omega_1}{\omega_2} = \frac{3}{4}$ .Find the first two degenerate energy levels. What can one say about the degeneracy of energy levels when the ratio between ω_1 and ω_2 is not a rational number.

7. A particle of mass m is confined to move in the potential $\left(m\omega^2 x^2\right)/2$. Its normalized wave function is

$$\psi(x) = \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-\left(\beta x^2/2\right)}$$

where β is a constant of appropriate dimension.
(a) Obtain a dimensional expression for β in terms of m, ω and $\hbar$.
(b) It can be shown that the above wave function is the linear combination

$$\psi(x) = a\psi_0(x) + b\psi_2(x)$$

where $\psi_0(x)$ is the normalized ground state wave function and $\psi_2(x)$ is the normalized second excited state wave function of the potential. Evaluate b and hence calculate the expectation value of the energy of the particle in this state $\psi(x)$.
Given: $I_0(\beta) = \int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$, $\quad I_n(\beta) = \int_{-\infty}^{+\infty} \left(x^2\right)^n e^{-\beta x^2} dx = (-1)^n \frac{\partial^n}{\partial \beta^n}\left(I_0(\beta)\right)$,
$\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{\frac{-\beta x^2}{2}}$

8. Consider an 3D isotropic harmonic oscillator show that the degeneracy g_n of the nth excited state, which is equal the number of ways the non negative integers n_x, n_y, n_z may be chosen to total to n, is given by

$$g_n = \frac{1}{2}(n+1)(n+2)$$

9. * A charged particle of mass ' m ' and charge ' q ' is bound in a 1-dimensional simple harmonic oscillator potential of angular frequency ' ω '. An electric field E_0 is turned on.
(a) What is the total potential V(x) experienced by the charge ?
(b) Express the total potential in the form of an effective harmonic oscillator potential.
(c) Sketch V (x) versus x.
(d) What is the ground state energy of the particle in this potential?
(e) What is the expectation value of the position (x) if the charge is in its ground state ?

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Tutorial Sheet 11

* marked problems will be solved in the tutorial class (D3-D4: Wednesday, D1-D2: Saturday)

Statistical Mechanics:

1. A national powerball lottery uses two sets of balls. The first set consists of 59 sequentially numbered balls and the second set consists of 35 sequentially numbered balls. Assume equal probability of choosing any ball and that all the balls are differently numbered. Five balls are chosen without replacement from the set of 59. Then one ball is chosen from the set of 35. Calculate the number of ways these six balls can be chosen (and thus your probability of winning the grand powerball prize).

2. Suppose we have 20 coins and we flip all of them together.

   (a) Considering all the coins to be independent of each other, how many possible outcomes (no. of microstates) do you expect with such a flipping?

   (b) How many ways are there for obtaining 12 heads and 8 tails?

   (c) What is the probability of obtaining 12 heads and 8 tails regardless of the order? They are called macrostates.

3. Three indistinguishable particles (say electrons) are to be arranged in three different energy levels of energy 0, E and $2E$, with respective degeneracies (ignore spin degeneracy) 2, 10 and 20. The total energy available is $3E$. What are the possible distributions and what are their probabilities?

4. * Consider a particle confined to a 3D harmonic oscillator potential, $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + 4z^2)$

   (a) Calculate the ground state energy of the particle.

   (b) What is the degeneracy of the state with energy, $E = 7\hbar\omega$?

5. A certain thermodynamic system has non-degenerate energy levels, with energies 0, E, $3E$, $5E$ and $9E$. Suppose that there are four particles, with total energy $U = 9E$. Identify the possible distribution of particles and evaluate their microstates when (a) the particles are distinguishable, (b) the particles are identical bosons and (c) the particles are identical fermions.

6. In how many ways three electrons can occupy ten states (include spin degeneracy)? Is the number same as the way in which three persons can occupy ten chairs in a room? State the reason. In case the number is different, find the other number also.

7. The energy of a particle in a 3-D cubical box is given by

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

If these levels are going to be occupied by electrons, write the energy values corresponding to the five lowest levels, taking into account the spin degeneracy. If three electrons occupy these states, find out the possible distributions which would yield a total energy of $18\pi^2\hbar^2/2mL^2$. Also find out the probability for each distributions.

A system has one state with energy 0, four states with energy $2E$ and eight states with energy $3E$. Six electrons are to be distributed among these states such that their total energy is $12E$. Consider a configuration (j, m, n) in which j electrons are in 0 energy state, m electrons are in $2E$ energy state and n electrons are in $3E$ state.

(a) Calculate the total number of microstates for the configuration (1,3,2).

(b) Find the ratio of probability of occurrence of a configuration (2,0,4) to that of a configuration (1,3,2)

8. * Consider a system of five particles trapped in a 1-D harmonic oscillator potential.

(a) What are the microstates of the ground state of this system for classical particles, identical Bosons and identical spin half Fermions.

(b) Suppose that the system is excited and has one unit of energy ($\hbar\omega$) above the corresponding ground state energy in each of the three cases. Calculate the number of microstates for each of the three cases.

(c) Suppose that the temperature of this system is low, so that the total energy is low (but above the ground state), describe in a couple of sentences, the difference in the behavior of the system of identical bosons from that of the system of classical particles.