PH-107: Introduction to Quantum Mechanics
Tutorial Sheet 3

* marked problems will be solved in the Wednesday tutorial class

Wave packets: Group and Phase Velocity

1. Consider two wave functions $\psi_1(y,t) = 5y\cos 7t$ and $\psi_2(y,t) = -5y\cos 9t$, where y and t are in meters and seconds, respectively. Show that their superposition generates a wave packet. Plot it and identify the modulated and modulating functions.

2. *Two harmonic waves which travel simultaneously along a wire are represented by

$$y_1 = 0.002\cos(8.0x - 400t) \quad \& \quad y_2 = 0.002\cos(7.6x - 380t)$$

where x, y are in meters and t is in sec.

   (a) Find the resultant wave and its phase and group velocities

   (b) Calculate the range Δx between the zeros of the group wave. Find the product of Δx and Δk ? [Ans.: $v_p = 50$ m/s, $v_g = 50$ m/s, $\Delta x = 5\pi$ m, $\Delta x \Delta k = 2\pi$]

3. The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{gk + Tk^3/\rho}$, where g is the acceleration due to gravity, ρ is the density of the liquid, and T is the surface tension (which gives an upward force on an element of the surface liquid). Find the phase and group velocities for the limiting cases when the surface waves have:
(a) very large wavelengths and
(b) very small wavelengths.

4. *Calculate the group and phase velocities for the wave packet corresponding to a relativistic particle.

5. Consider an electromagnetic (EM) wave of the form $A\exp(i[kx - \omega t])$. Its speed in free space is given by $c = \frac{\omega}{k} = 1/\sqrt{\epsilon_0\mu_0}$, where ϵ_0, μ_0 is the electric permittivity, magnetic permeability of free space, respectively.

   (a) Find an expression for the speed (v) of EM waves in a medium, in terms of its permittivity ε and permeability μ.

   (b) Suppose the permittivity of the medium depends on the frequency, given by $\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ where ω_p is a constant called the plasma frequency, find the dispersion relation for the EM waves in a medium. wp is a constant and is called the plasma frequency of the medium (assume $\mu = \mu_0$).

(c) Consider waves with $\omega = 3\omega_p$. Find the phase and group velocity of the waves. What is the product of group and phase velocities?

6. The dispersion relation for a lattice wave propagating in a 1-D chain of atoms of mass m bound together by a force constant β is given by $\omega = \omega_0 \sin\left(\frac{ka}{2}\right)$, where $\omega_0 = \sqrt{4\beta/m}$ and a is the distance between the atoms.

   (a) Show that the medium is non- dispersive in the long wavelength limits.

   (b) Find the group and phase velocities at $k = \pi/a$. [Ans.: 0, $\omega_o a/\pi$]

7. *Consider a squre 2-D system with small balls (each of mass m) connected by springs. The spring constants along the x- and y-directions are β_x and β_y, respectively. The dispersion relation for this system is given by

$$-\omega^2 m + 2\beta_x (1 - \cos k_x a_x) + 2\beta_y (1 - \cos k_y a_y) = 0$$

where $\vec{k} = k_x\hat{i} + k_y\hat{j}$ is the wave vector and a_x,a_y are the natural distances between the two successive masses along the x-, y-directions, respectively. Find the group velocity and the angle that it makes with the x-axis